Formulas for phase recovering from phaseless scattering data at fixed frequency

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Abstract. We consider quantum and acoustic wave propagation at fixed frequency for compactly supported scatterers in dimension $d \geq 2$. In these framework we give explicit formulas for phase recovering from appropriate phaseless scattering data. As a corollary, we give global uniqueness results for quantum and acoustic inverse scattering at fixed frequency without phase information.

1. Introduction

We consider the equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad E > 0, \tag{1.1}$$

where Δ is the Laplacian, v is a scalar potential such that

$$\begin{aligned} &v \in L^\infty(\mathbb{R}^d), \quad supp\, v \subset D, \\ &D \ \text{ is an open bounded domain in } \ \mathbb{R}^d. \end{aligned} \tag{1.2}$$

Equation (1.1) can be considered as the quantum mechanical Schrödinger equation at fixed energy E.

Equation (1.1) can also be considered as the acoustic equation at fixed frequency ω. In this setting

$$E = \big(\frac{\omega}{c_0}\big)^2, \quad v(x) = (1 - n^2(x))\big(\frac{\omega}{c_0}\big)^2, \tag{1.3}$$

where c_0 is a reference sound speed, $n(x)$ is a scalar index of refraction.

For equation (1.1) we consider the classical scattering solutions ψ^+ continuous and bounded on $\mathbb{R}^d$ and specified by the following asymptotics as $|x| \to \infty$:

$$\begin{aligned} &\psi^+(x,k) = e^{ikx} + c(d,|k|)\frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f(k, |k|\frac{x}{|x|}) + O\big(\frac{1}{|x|^{(d+1)/2}}\big), \\ &x \in \mathbb{R}^d, \ k \in \mathbb{R}^d, \ k^2 = E, \ c(d,|k|) = -\pi i(-2\pi i)^{(d-1)/2}|k|^{(d-3)/2}, \end{aligned} \tag{1.4}$$

where a priori unknown function $f = f(k,l)$, $k, l \in \mathbb{R}^d$, $k^2 = l^2 = E$, arising in (1.4) is the classical scattering amplitude for (1.1).

In order to find ψ^+ and f from v one can use the following Lippmann-Schwinger integral equation (1.5) and formula (1.7) (see, e.g., [BS], [FM]):

$$\psi^+(x,k) = e^{ikx} + \int\limits_D G^+(x-y,k)v(y)\psi^+(y,k)dy, \tag{1.5}$$

$$G^+(x,k) \stackrel{\text{def}}{=} -(2\pi)^{-d} \int\limits_{\mathbb{R}^d} \frac{e^{i\xi x}d\xi}{\xi^2 - k^2 - i0} = G_0^+(|x|,|k|), \tag{1.6}$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{R}^d$, $k^2 = E$, and G_0^+ depends also on d;

$$f(k,l) = (2\pi)^{-d} \int\limits_D e^{-ily} v(y) \psi^+(y,k) dy, \tag{1.7}$$

where $k \in \mathbb{R}^d$, $l \in \mathbb{R}^d$, $k^2 = l^2 = E$.

We recall that ψ^+ describes scattering of the incident plane waves e^{ikx} on the potential v. And the second term of the right-hand side of (1.4) describes the scattered spherical waves.

In addition to ψ^+, we consider also the function R^+ describing scattering of spherical waves generated by point sources. The function $R^+ = R^+(x,x',E)$, $x \in \mathbb{R}^d$, $x' \in \mathbb{R}^d$, can be defined as the Schwartz kernel of the standard resolvent $(-\Delta + v - E - i0)^{-1}$. Note that $R^+ = R^+(x,x',E) = -G_0^+(|x-x'|, \sqrt{E})$ for $v \equiv 0$, where G_0^+ is the function of (1.6). Given v, to determine R^+ one can use, in particular, the following integral equation

$$R^+(x,x',E) = -G_0^+(|x-x'|, \sqrt{E}) + \int\limits_D G_0^+(|x-y|, \sqrt{E}) v(y) R^+(y,x',E) dy, \tag{1.8}$$

where $x \in \mathbb{R}^d$, $x' \in \mathbb{R}^d$.

The function $R^+(x,x',E)$ at fixed $x' \in \mathbb{R}^d$ describes scattering of the spherical wave $-G_0^+(|x-x'|, \sqrt{E})$ generated by a point source at x'. In addition,

$$\begin{aligned} &R^+(x,x',E) = -\frac{c(d,\sqrt{E})}{(2\pi)^d} \frac{e^{i\sqrt{E}|x|}}{|x|^{(d-1)/2}} \psi^+\big(x', -\sqrt{E}\frac{x}{|x|}\big) + \\ &O\big(\frac{1}{|x|^{(d+1)/2}}\big) \ \text{ as } \ |x| \to \infty \ \text{ at fixed } \ x', \end{aligned} \tag{1.9}$$

$$R^+(x,x',E) = R^+(x',x,E), \tag{1.10}$$

where c is the constant of (1.4), ψ^+ is the function of (1.4), (1.5).

In connection with aforementioned facts concerning R^+ see, e.g., Section 1 of Chapter IV of [FM].

Let

$$\mathbb{S}_r^{d-1} = \{k \in \mathbb{R}^d : \ |k| = r\}, r > 0. \qquad (1.11).$$

We consider the following three types of scattering data for equation (1.1):

(a) $f(k,l)$, where $(k,l) \in \Omega_f' \subseteq \Omega_f$,

$$\Omega_f = \mathbb{S}_{\sqrt{E}}^{d-1} \times \mathbb{S}_{\sqrt{E}}^{d-1}; \tag{1.12a}$$

(b) $\psi^+(x,k)$, where $(x,k) \in \Omega_\psi' \subseteq \Omega_\psi$,

$$\Omega_\psi = (\mathbb{R}^d \backslash (D \cup \partial D)) \times \mathbb{S}_{\sqrt{E}}^{d-1}, \tag{1.12b}$$

assuming (1.13);

(c) $R^+(x,y,E)$, where $(x,y)\in\Omega'_R\subseteq\Omega_R$,

$$\Omega_R = (\mathbb{R}^d\backslash(D\cup\partial D))\times(\mathbb{R}^d\backslash(D\cup\partial D)), \tag{1.12c}$$

assuming (1.13).

In (1.12b), (1.12c) we assume also that

$$\mathbb{R}^d\backslash(D\cup\partial D)\ \text{ is connected.} \tag{1.13}$$

We consider the following inverse scattering problems for equation (1.1) at fixed E:

Problem 1.1a. Reconstruct potential v on $\mathbb{R}^d$ from its scattering amplitude f on some appropriate $\Omega'_f\subseteq\Omega_f$.

Problem 1.1b. Reconstruct potential v on $\mathbb{R}^d$ from its scattering data ψ^+ on some appropriate $\Omega'_\psi\subseteq\Omega_\psi$.

Problem 1.1c. Reconstruct potential v on $\mathbb{R}^d$ from its scattering data R^+ on some appropriate $\Omega'_R\subseteq\Omega_R$.

Problem 1.2a. Reconstruct potential v on $\mathbb{R}^d$ from its phaseless scattering data $|f|^2$ on some appropriate $\Omega'_f\subseteq\Omega_f$.

Problem 1.2b. Reconstruct potential v on $\mathbb{R}^d$ from its phaseless scattering data $|\psi^+|^2$ on some appropriate $\Omega'_\psi\subseteq\Omega_\psi$.

Problem 1.2c. Reconstruct potential v on $\mathbb{R}^d$ from its phaseless scattering data $|R^+|^2$ on some appropriate $\Omega'_R\subseteq\Omega_R$.

Note that in quantum mechanical scattering experiments in the framework of model described by equation (1.1) the phaseless scattering data $|f|^2$, $|\psi^+|^2$, $|R^+|^2$ of Problems 1.2a-1.2c can be measured directly, whereas the complete scattering data f, ψ^+, R^+ of Problems 1.1a-1.1c are not accessible for direct measurements. Therefore, Problems 1.2 are of particular interest in the framework of quantum mechanical inverse scattering.

As regards to acoustic scattering experiments in the framework of the model described by (1.1), (1.3), the complete scattering data f, ψ^+, R^+ of Problems 1.2 can be measured directly. Nevertheless, in some cases it may be more easy to measure the phaseless versions of these data. Therefore, Problems 1.2 are also of interest in the framework of acoustic inverse scattering.

On the other hand, in the literature many more results are given on Problems 1.1 (see [ABR], [Be], [Bu], [BAR], [BSSR], [ChS], [E], [F1], [G], [HH], [HN], [I], [IN], [M], [Na], [N1]-[N6], [R], [S] and references therein) than on Problems 1.2 (see Chapter X of [ChS] and recent works [KR], [N7] and references therein).

The works [K1], [K2], [K3], [KR], [N7] give also results on analogs of Problems 1.2, where E is not fixed. Besides, analogs of Problems 1.2 in dimension $d=1$, where E is not fixed, were considered, in particular, in [AS], [KS].

Let

$$\mathcal{B}_r=\{x\in\mathbb{R}^d:\ |x|<r\},\quad r>0. \tag{1.14}$$

Suppose that, for some $r>0$,

$$D\subseteq\mathcal{B}_r. \tag{1.15}$$

In connection with Problems 1.1, under assumption (1.15), it is well known, in particular, that any of the scattering data (a) f on Ω_f, (b) ψ^+ on $\partial\mathcal{B}_r \times \mathbb{S}^{d-1}_{\sqrt{E}}$, or (c) R^+ on $\partial\mathcal{B}_r \times \partial\mathcal{B}_r$ uniquely and constructively determine two other data; see [Be].

In addition, in [N1] for $d \geq 3$ (see also [N3]) and in [Bu] for $d = 2$ it was shown that f on Ω_f at fixed E uniquely and constructively determines v on $\mathbb{R}^d$, under assumption (1.2). For related exact stability estimates, see [S], [HH], [IN], [I]. Besides, for approximate but efficient methods for solving Problem 1.1a, see [N4], [N5], [ABR], [BAR], [N6].

The main results of the present work can be summarized as follows.

First, we give explicit asymptotic formulas for finding $f(k,l)$ at fixed $(k,l) \in \Omega_f$, $k \neq l$, from $|\psi^+(x,k)|^2$ for $x = sl/|l|$, $s \in \Lambda = [r_1, +\infty[$ for arbitrary large $r_1 \geq r$ (assuming, e.g., (1.15)); see Theorem 2.1 and Corollary 2.1 of Section 2.

In addition, we have the asymptotic formula (2.11) for finding $|\psi^+(x',k)|^2$ at fixed $(x',k) \in \Omega_\psi$ from $|R^+(x,x',E)|^2$ for $x = -sk/|k|$, $s \in \Lambda$ for any unbounded $\Lambda \subset [r, +\infty[$ (assuming, e.g., (1.15)).

The aforementioned formulas give explicit reductions of Problems 1.2b, 1.2c to Problem 1.1a for appropriate Ω'_ψ, Ω'_R and Ω'_f.

In connection with reductions of Problems 1.2b, 1.2c to Problem 1.1a we give also additional global uniqueness results summarized in Theorem 2.2 of Section 2.

Second, we give global uniqueness results for Problem 1.2b for the case when Ω'_ψ is an open subset of Ω_ψ and for Problem 1.2c for the case when Ω'_R is an open subset of Ω_R; see Theorem 2.3 of Section 2. In this connection we recall also that for Problem 1.1a in its initial formulation there is no uniqueness, in general; see [N7].

Actually, as soon as Problems 1.2b, 1.2c are reduced to Problem 1.1a, one can use all known results for exact or approximate solving Problem 1.1a; see [N6] and other works cited above in connection with Problems 1.1.

Finally, we indicate some possible generalizations and extentions of results of the present work; see Remarks 2.1-2.4 at the end of Section 2. In particular, in a subsequent work we plan to consider phaseless inverse scattering in dimension $d = 1$ when E is not fixed, using an analog of Theorem 2.1 for $d = 1$.

To our knowledge, no exact general result on phase recovering from phaseless scattering data for equation (1.1) at fixed E was given in the literature before the present work.

The main results of the present work are presented in detail in the next section.

2. Main results

We represent f and c of (1.4) as follows:

$$\begin{aligned} f(k,l) &= |f(k,l)|e^{i\alpha(k,l)}, \\ c(d,|k|) &= |c(d,|k|)|e^{i\beta(d,|k|)}. \end{aligned} \tag{2.1}$$

We consider

$$a(x,k) = |x|^{(d-1)/2}(|\psi^+(x,k)|^2 - 1), \tag{2.2}$$

$$a_0(x,k) = 2Re\,\big(c(d,|k|)e^{i(|k||x|-kx)}f\big(k,|k|\frac{x}{|x|}\big)\big), \tag{2.3}$$

$$\delta a(x,k) = a(x,k) - a_0(x,k) \tag{2.4}$$

for $x \in \mathbb{R}^d\backslash\{0\},\ \ k \in \mathbb{R}^d\backslash\{0\}$, where ψ^+ are the scattering solutions of (1.4), (1.5), f is the scattering amplitude of (1.4), (1.7), c is the constant of (1.4).

For real-valued potential v satisfying (1.2) one can show that

$$|\delta a(x,k)| \le \delta_0(|x|,|k|), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{R}^d\backslash\{0\}, \tag{2.5}$$

$$\begin{aligned} \delta_0(r,\rho) &= O(r^{-1/2}) \ \text{ for } \ r \to +\infty,\ d=2, \\ \delta_0(r,\rho) &= O(r^{-1}) \ \text{ for } \ r \to +\infty,\ d \ge 3, \end{aligned} \tag{2.6}$$

at fixed $\rho > 0$ (where δ_0 depends also on v). Estimates (2.5), (2.6) are proved in Section 3.

The key result of the present work consists in the following theorem:

Theorem 2.1. *Let real-valued potential v satisfy (1.2), $d \ge 2$, and f, a be the functions of (1.4), (2.2). Let $(k,l) \in \Omega_f$ of (1.12a), $k \ne l$, and*

$$T = 2\pi\big(E^{1/2}\big(1 - \frac{kl}{E}\big)\big)^{-1}. \tag{2.7}$$

Then the following formulas hold:

$$\begin{aligned} &|f|\begin{pmatrix}\cos\alpha \\ \sin\alpha\end{pmatrix} = (2|c|\sin(2\pi T^{-1}(s_1 - s_2)))^{-1}\times \\ &\begin{pmatrix} -\sin(2\pi T^{-1}s_2 + \beta) & \sin(2\pi T^{-1}s_1 + \beta) \\ -\cos(2\pi T^{-1}s_2 + \beta) & \cos(2\pi T^{-1}s_1 + \beta)\end{pmatrix}\times \\ &\left(\begin{pmatrix} a((s_1+nT)l/|l|,k) \\ a((s_2+nT)l/|l|,k)\end{pmatrix} - \begin{pmatrix} \delta a((s_1+nT)l/|l|,k) \\ \delta a((s_2+nT)l/|l|,k)\end{pmatrix}\right), n \in \mathbb{N}, \end{aligned} \tag{2.8}$$

$s_1, s_2 \in [0,T]$, $s_1 \ne s_2$ *(mod$\,T/2$), $\alpha = \alpha(k,l)$, $|f| = |f(k,l)|$, $|c| = |c(d,|k|)|$, $\beta = \beta(d,|k|)$, where α, β are the angles of (2.1), c is the constant of (1.4), δa is defined by (2.4),*

$$|\delta a(s_j + nT)l/|l|,k)| \le \delta_0(s_j + nT, |k|) = \left\{ \begin{array}{c} O\big((nT)^{-1/2}\big),\ n \to \infty,\ d=2, \\ O\big((nT)^{-1}\big),\ n \to \infty,\ d \ge 3, \end{array}\right. \tag{2.9}$$

where $j=1,2$, δ_0 *is the function of (2.5).*

Theorem 2.1 is proved in Section 3.

In connection with formula (2.8) we consider

$$\Lambda = \cup_{j=1}^{+\infty}(s_1+n_jT)\cup(s_2+n_jT),\ n_j\in\mathbb{N},\ n_j\to+\infty,\ j\to+\infty, \tag{2.10}$$

for fixed $T>0$ and $s_1,s_2\in[0,\tau]$, $s_1\neq s_2\,(mod\,T/2)$.

Corollary 2.1. *Let real-valued potential* v *satisfy (1.2),* $d\geq 2$*, and* ψ^+*,* f *be the scattering functions of (1.4). Let* $(k,l)\in\Omega_f$ *of (1.12a),* $k\neq l$*, and* T *be defined by (2.7). Then:*

$|\psi^+(sl/|l|,k)|^2$*,* $s\in\Lambda$ *of (2.10), uniquely determines* $f(k,l)$ *via (2.1), (2.8), (2.9).*

In addition, due to (1.9), we have that

$$|\psi^+(x',k)|^2 = (2\pi)^d|c(d,|k|)|^{-1}s^{(d-1)/2}|R^+(-sk/|k|,x',k^2)|^2+O(s^{-1}) \tag{2.11}$$

for $s\to+\infty$ at fixed $x'\in\mathbb{R}^d$, $k\in\mathbb{R}^d\backslash\{0\}$.

Formulas (2.2), (2.8),(2.9), (2.11) give explicit reductions of Problems 1.2b, 1.2c to Problem 1.1a for appropriate Ω'_ψ, Ω'_R and Ω'_f. In this connection we have also the uniqueness results of the following theorem:

Theorem 2.2. *Let real-valued potential* v *satisfy (1.2), (1.13), (1.15),* $d\geq 2$*, and* ψ^+*,* f*,* R^+ *be the scattering functions of (1.4), (1.5), (1.7), (1.8), (1.9). Then:*

(1) the scattering amplitude $f(k,l)$ *for fixed* $(k,l)\in\Omega_f$*,* $k\neq l$*, is uniquely determined by the phaseless scattering data* $|\psi^+(sl/|l|,k)|^2$*,* $s\in[r_1,r_2]$*,* $r\leq r_1<r_2$*,*

(2) the scattering amplitude $f(k,l)$ *for fixed* $(k,l)\in\Omega_f$*,* $k\neq l$*, is uniquely determined by the phaseless scattering data*

$$|R^+(-sk/|k|,s'l/|l|,E)|^2,\ \ (s,s')\in[r_1,r_2]\times[r_1',r_2'],\ r\leq r_1<r_2,\ r\leq r_1'<r_2',$$

where E *and* r *are the parameters of (1.12a), (1.15);*

(3) the phaseless scattering data $|\psi^+|^2$ *on a fixed open subset* Ω'_ψ *of* Ω_ψ *uniquely determine the scattering amplitude* f *on* Ω_f *at fixed* E*,*

(4) the phaseless scattering data $|R^+|^2$ *on a fixed open subset* Ω'_R *of* Ω_R *uniquely determine the scattering amplitude* f *on* Ω_f *at fixed* E*.*

Theorem 2.2 is proved in Section 3. This proof involves real-analytic continuations.

As a corollary of results of [N1], [N3], [Bu] and items (3) and (4) of Theorem 2.2, we have also the following global uniqueness results on Problems 1.2b, 1.2c:

Theorem 2.3. *Let real-valued potential* v *satisfy (1.2), (1.13),* $d\geq 2$*. Then:*

- *the phaseless scattering data* $|\psi^+|^2$ *on a fixed open subset* Ω'_ψ *of* Ω_ψ *uniquely determine* v *in* $L^\infty(\mathbb{R}^d)$*,*
- *the phaseless scattering data* $|R^+|^2$ *on a fixed open subset* Ω'_R *of* Ω_R *uniquely determine* v *in* $L^\infty(\mathbb{R}^d)$*.*

Remark 2.1. In all aforementioned results of this section the assumption that v is real-valued can be replaced by the assumption that v is complex-valued and equation (1.5) is uniquely solvable for $\psi^+\in L^\infty(D)$ at fixed $k\in\mathbb{R}^d$, $k^2=E>0$.

Remark 2.2. In Problems 1.1, 1.2 the assumption that v of (1.1) is compactly supported (supported in D) can be replaced by the assumption that v has sufficient decay at infinity; see e.g. [ChS], [E], [F1], [G], [HN], [N1]-[N6], [R], [VW], [W], [WY]. In this case, especially in Problems 1.1b, 1.1c, 1.2b, 1.2c, it is natural to assume that v is a priori known on $\mathbb{R}^d\backslash D$. Theorem 2.1, Corollary 2.1 and formula (2.11) remain valid for v with sufficient decay at infinity.

Remark 2.3. Theorem 2.1, Corollary 2.1 and formula (2.11) have analogs in dimension $d = 1$. Using these results, in a subsequent work we plan to consider phaseless inverse scattering in dimension $d = 1$ when frequency is not fixed.

Remark 2.4. The approach of the present work can be also used for phaseless inverse scattering for obstacles.

3. Proofs of estimates (2.5), (2.6) and Theorem 2.1

3.1. Proofs of estimates (2.5), (2.6). Note that the asymptotic formula (1.4) holds uniformly in $x/|x| \in \mathbb{S}^{d-1}$. Therefore, we have that

$$\psi^+(x,k) = \psi_1^+(x,k) + \delta\psi^+(x,k), \tag{3.1}$$

$$\psi_1^+(x,k) = e^{ikx} + c(d,|k|)\frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f(k,|k|\frac{x}{|x|}), \tag{3.2}$$

$$|\delta\psi^+(x,k)| \le \delta_1(|x|,|k|), \tag{3.3}$$

$$\delta_1(r,\rho) = O\big(r^{-(d+1)/2}\big) \ \text{ for } \ r \to +\infty \ \text{ at fixed } \ \rho > 0, \tag{3.4}$$

where $x \in \mathbb{R}^d$, $k \in \mathbb{R}^d\backslash\{0\}$.

Using (2.3), (3.1), (3.2) we obtain that

$$\begin{aligned}
&|\psi^+(x,k)|^2 = \psi^+(x,k)\overline{\psi^+(x,k)} = \\
&1 + |x|^{-(d-1)/2}a_0(x,k) + |x|^{-(d-1)}|c(d,|k|)|^2|f(k,|k|x/|x|)|^2 + \\
&2Re\,(\delta\psi^+(x,k)\overline{\psi_1^+(x,k)}) + |\delta\psi^+(x,k)|^2.
\end{aligned} \tag{3.5}$$

Due to (2.2)-(2.4), (3.5), we have that

$$\begin{aligned}
&\delta a(x,k) = |x|^{-(d-1)/2}|c(d,|k|)|^2|f(k,|k|x/|x|)|^2 + \\
&2|x|^{(d-1)/2}Re\,(\delta\psi^+(x,k)\overline{\psi_1^+(x,k)}) + |x|^{(d-1)/2}|\delta\psi^+(x,k)|^2.
\end{aligned} \tag{3.6}$$

Note that

$$|f(k,l)| \le C_f(\sqrt{E}), \quad (k,l) \in \Omega_f, \tag{3.7}$$

for some positive $C_f(\sqrt{E})$, where Ω_f is defined by (1.12a).

Using (3.3), (3.6), (3.7) we obtain (2.5) with $\delta_0(r,\rho)$ given by

$$\begin{aligned}
&\delta_0(r,\rho) = \delta_{0,1}(r,\rho) + \delta_{0,2}(r,\rho) + \delta_{0,3}(r,\rho), \\
&\delta_{0,1}(r,\rho) = r^{-(d-1)/2}|c(d,\rho)|^2(C_f(\rho))^2, \\
&\delta_{0,2}(r,\rho) = 2r^{(d-1)/2}\delta_1(r,\rho)(1 + c(d,\rho)r^{-(d-1)/2}C_f(\rho), \\
&\delta_{0,3}(r,\rho) = r^{(d-1)/2}(\delta_1(r,\rho))^2,
\end{aligned} \tag{3.8}$$

where $r > 0$, $\rho > 0$.

Formulas (3.4), (3.8) imply (2.6). This completes the proof of estimates (2.5), (2.6).

3.2. Proof of Theorem 2.1. Due to (2.1), (2.3), (2.7) we have that

$$\begin{aligned}&a_0(sl/|l|,k) = 2|c(d,|k|)||f(k,l)|\cos(2\pi T^{-1}s + \alpha(k,l) + \beta(d,|k|)),\\ &(k,l) \in \Omega_f, \quad k \neq l, \quad s > 0,\end{aligned} \tag{3.9}$$

where Ω_f is defined by (1.12a), T is defined by (2.7). In addition, one can see that

$$T > 0 \ \text{ for } \ (k,l) \in \Omega_f, \quad k \neq l. \tag{3.10}$$

Due to (2.4), (3.9), we have that

$$\begin{aligned}&|f|\big(\cos(2\pi T^{-1}s + \beta)\cos\alpha - \sin(2\pi T^{-1}s + \beta)\sin\alpha\big) =\\ &(2|c|)^{-1}(a(sl/|l|,k) - \delta a(sl/|l|,k)), \quad (k,l) \in \Omega_f, \quad k \neq l, \quad s > 0,\end{aligned} \tag{3.11}$$

$\alpha = \alpha(k,l)$, $|f| = |f(k,l)|$, $|c| = |c(d,|k|)|$, $\beta = \beta(d,|k|)$. Using (3.11) for $s = s_1 + nT$ and $s = s_2 + nT$, where $s_1, s_2 \in [0,T]$, $n \in \mathbb{N}$, we obtain the system

$$\begin{aligned}&\begin{pmatrix}\cos(2\pi T^{-1}s_1 + \beta) & -\sin(2\pi T^{-1}s_1 + \beta)\\ \cos(2\pi T^{-1}s_2 + \beta) & -\sin(2\pi T^{-1}s_2 + \beta)\end{pmatrix}|f|\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix} =\\ &(2|c|)^{-1}\begin{pmatrix}a((s_1+nT)l/|l|,k) - \delta a((s_1+nT)l/|l|,k)\\ a((s_2+nT)l/|l|,k) - \delta a((s_2+nT)l/|l|,k)\end{pmatrix}.\end{aligned} \tag{3.12}$$

Formula (2.8) follows from (3.12). Formula (2.9) follows from (2.5), (2.6).

Theorem 2.1 is proved.

4. Proof of Theorem 2.2

Note that

$$|\psi^+|^2 = \psi^+\overline{\psi^+}, \tag{4.1}$$

$$|R^+|^2 = R^+\overline{R^+}. \tag{4.2}$$

Note also that

$$\mathbb{R}^d\backslash(\mathcal{B}_r \cup \partial\mathcal{B}_r) \subseteq \mathbb{R}^d\backslash(D \cup \partial D) \tag{4.3}$$

under assumption (1.15).

4.1. Proof of item (1). Due to equation (1.1) for ψ^+ and assumptions (1.2), we have that

$$-\Delta\psi^+(x,k) = E\psi(x,k), \quad x \in \mathbb{R}^d\backslash(D \cup \partial D), \quad \text{for each} \quad k \in \mathbb{S}^{d-1}_{\sqrt{E}}. \tag{4.4}$$

Therefore,

$$\begin{aligned}&\psi^+(\cdot,k) \ \text{ is (complex − valued) real − analytic on}\\ &\mathbb{R}^d\backslash(D \cup \partial D) \ \text{ at fixed } \ k \in \mathbb{S}^{d-1}_{\sqrt{E}}.\end{aligned} \tag{4.5}$$

Using (4.1), (4.4) we obtain that

$$|\psi^+(\cdot,k)|^2 \ \text{ is real} - \text{analytic on } \ \mathbb{R}^d\backslash(D\cup\partial D) \ \text{ at fixed } \ k\in\mathbb{S}^{d-1}_{\sqrt{E}}. \tag{4.6}$$

As a corollary of (4.3), (4.6), we have that

$$|\psi^+(sl/|l|,k)|^2 \ \text{ is real} - \text{analytic in } \ s\in]r,+\infty[\ \text{ at fixed } \ (k,l)\in\Omega_f. \tag{4.7}$$

Therefore, at fixed $(k,l)\in\Omega_f$, the function $|\psi^+(sl/|l|,k)|^2$ given for all $s\in[r_1,r_2]$, where $r\le r_1<r_2$, uniquely determines this function for all $s\in]r,+\infty[$ via real-analytic continuation. This result and Corollary 2.1 imply item (1) of Theorem 2.2.

4.2. Proof of item (2). Note that

$$\begin{aligned}(-\Delta_x+v(x)-E)R^+(x,x',E)&=\delta(x-x'),\\(-\Delta_{x'}+v(x)-E)R^+(x,x',E)&=\delta(x-x'),\end{aligned} \tag{4.8}$$

where $x\in\mathbb{R}^d$, $x'\in\mathbb{R}^d$, $E>0$, δ is the Dirac delta-function. In addition, due to (1.2), (4.8), we have that

$$\begin{aligned}(-\Delta_x-E)R^+(x,x',E)&=\delta(x-x'),\ x\in\mathbb{R}^d\backslash(D\cup\partial D),\ x'\in\mathbb{R}^d,\\(-\Delta_{x'}-E)R^+(x,x',E)&=\delta(x-x'),\ x\in\mathbb{R}^d,\ x'\in\mathbb{R}^d\backslash(D\cup\partial D).\end{aligned} \tag{4.9}$$

Therefore,

$$\begin{aligned}&R^+(\cdot,x',E) \ \text{ is (complex} - \text{valued) real} - \text{analytic on}\\&\mathbb{R}^d\backslash(D\cup\partial D\cup\{x'\}) \ \text{ for fixed } \ x'\in\mathbb{R}^d,\\&R^+(x,\cdot,E) \ \text{ is (complex} - \text{valued) real} - \text{analytic on}\\&\mathbb{R}^d\backslash(D\cup\partial D\cup\{x'\}) \ \text{ for fixed } \ x\in\mathbb{R}^d.\end{aligned} \tag{4.10}$$

Using (4.2), (4.10) we obtain that

$$\begin{aligned}&|R^+(\cdot,x',E)|^2 \ \text{ is real} - \text{analytic on } \ \mathbb{R}^d\backslash(D\cup\partial D\cup\{x'\}) \ \text{ for fixed } \ x'\in\mathbb{R}^d,\\&|R^+(x,\cdot,E)|^2 \ \text{ is real} - \text{analytic on } \ \mathbb{R}^d\backslash(D\cup\partial D\cup\{x'\}) \ \text{ for fixed } \ x\in\mathbb{R}^d.\end{aligned} \tag{4.11}$$

As a corollary of (4.3), (4.11), we have, in particular, that

$$\begin{aligned}&|R^+(-sk/|k|,s'l/|l|,E)|^2 \ \text{ is real} - \text{analytic in } \ s\in]r,+\infty[\ \text{ if } \ l\ne -k\\&\text{and in } \ s\in]r,+\infty[\backslash s' \ \text{ if } \ l=-k \ \text{ at fixed } \ s'\in[0,+\infty[.\end{aligned} \tag{4.12}$$

If $l\ne -k$, then using (2.11), (4.12) one can see that the phaseless data of item (2) uniquely determine $|\psi^+(s'l/|l|,k)|^2$, $s'\in[r_1',r_2']$, via real-analytic continuation in s. In turn, the latter data uniquely determine $f(k,l)$ due to the result of item (1).

Therefore, it remains to consider the case when $l = -k$. In addition, in view of (1.10), we can assume that $r_2' \le r_2$. Under these conditions, using (2.11), (4.12) one can see that phaseless data of item (2) uniquely determine $|\psi^+(s'l/|l|, k)|^2$, $s' \in [r_1', r_2'[$, via real-analytic continuation in s. In turn, the latter data uniquely determine $f(k,l)$ due to the result of item (1).

This completes the proof of item (2) of Theorem 2.2.

4.3. Proof of item (3). Note that, under our assumptions,

$$\psi^+(x,\cdot) \ \text{ is (complex} - \text{valued) real} - \text{analytic on } \ \mathbb{S}^{d-1}_{\sqrt{E}} \ \text{ at fixed } \ x \in \mathbb{R}^d. \tag{4.13}$$

Actually, this result is well-known and follows from consideration of (1.5) with $k \in \mathbb{C}^d$, $k^2 = k_1^2 + \ldots + k_d^2 = E$.

Using (4.1), (4.13) we obtain that

$$|\psi^+(x,\cdot)|^2 \ \text{ is real} - \text{analytic on } \ \mathbb{S}^{d-1}_{\sqrt{E}} \ \text{ at fixed } \ x \in \mathbb{R}^d. \tag{4.14}$$

Let

$$\mathcal{A}_{k',\varepsilon} = \{k \in \mathbb{S}^{d-1}_{\sqrt{E}} : \ |k - k'| < \varepsilon\}, \ \ k' \in \mathbb{S}^{d-1}_{\sqrt{E}}, \ \ \varepsilon > 0, \tag{4.15}$$

$$\mathcal{B}_{x',\varepsilon} = \{x \in \mathbb{R}^d : \ |x - x'| < \varepsilon\}, \ \ x' \in \mathbb{R}^d, \ \ \varepsilon > 0. \tag{4.16}$$

Since Ω'_ψ is open in Ω_ψ, we can take $x' \in \mathbb{R}^d \backslash (D \cup \partial D)$, $k' \in \mathbb{S}^{d-1}_{\sqrt{E}}$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that $\mathcal{B}_{x',\varepsilon_1} \times \mathcal{A}_{k',\varepsilon_2} \subseteq \Omega'_\psi$. In addition, using (1.13), (4.6), (4.14) one can see that already $|\psi^+|^2$ on $\mathcal{B}_{x',\varepsilon_1} \times \mathcal{A}_{k',\varepsilon_2}$ uniquely determines $|\psi^+|^2$ on Ω_ψ via sequential real-analytic continuations in x and in k.

In turn, $|\psi^+|^2$ on Ω_ψ uniquely determines f for all $(k,l) \in \Omega_f$, $k \ne l$, due to Corollary 2.1. In addition, under our assumptions, f is continuous on Ω_f and therefore, is uniquely determined also for $k = l$.

This completes the proof of item (3) of Theorem 2.2.

4.4. Proof of item (4). Since Ω'_R is open in Ω_R, we can take $y \in \mathbb{R}^d \backslash (D \cup \partial D)$, $y' \in \mathbb{R}^d \backslash (D \cup \partial D)$, $\varepsilon > 0$, $\varepsilon' > 0$ such that $\mathcal{B}_{y,\varepsilon} \times \mathcal{B}_{y',\varepsilon'} \subseteq \Omega'_R$. In addition, using (1.13), (4.11) one can see that already $|R^+|^2$ on $\mathcal{B}_{y,\varepsilon} \times \mathcal{B}_{y',\varepsilon'}$ uniquely determines $|R^+(x,x',E)|^2$ for all $(x,x') \in \Omega_R$, $x \ne x'$, via sequential real-analytic continuations in x and in x'. In turn, the latter data uniquely determine $|\psi^+|^2$ on Ω_ψ via (2.11). Finally, due to item (3), $|\psi^+|^2$ on Ω_ψ uniquely determines f on Ω_f.

This completes the proof of item (4) of Theorem 2.2.

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